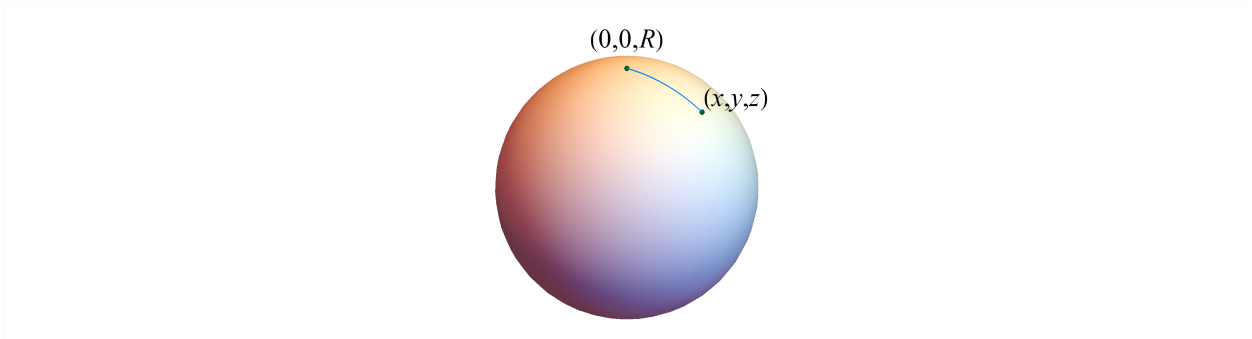


Math Boot Camp: Coordinate Systems

You can skip this boot camp if you can answer the following question:

Example

Staying on a sphere of radius R , what is the shortest distance between the point $(0, 0, R)$ on the z -axis and a general point (x, y, z) on the sphere?



2D Polar Coordinates

A generic 2D vector in Cartesian coordinates is

$$\vec{A} = A_x \hat{x} + A_y \hat{y} \quad (1)$$

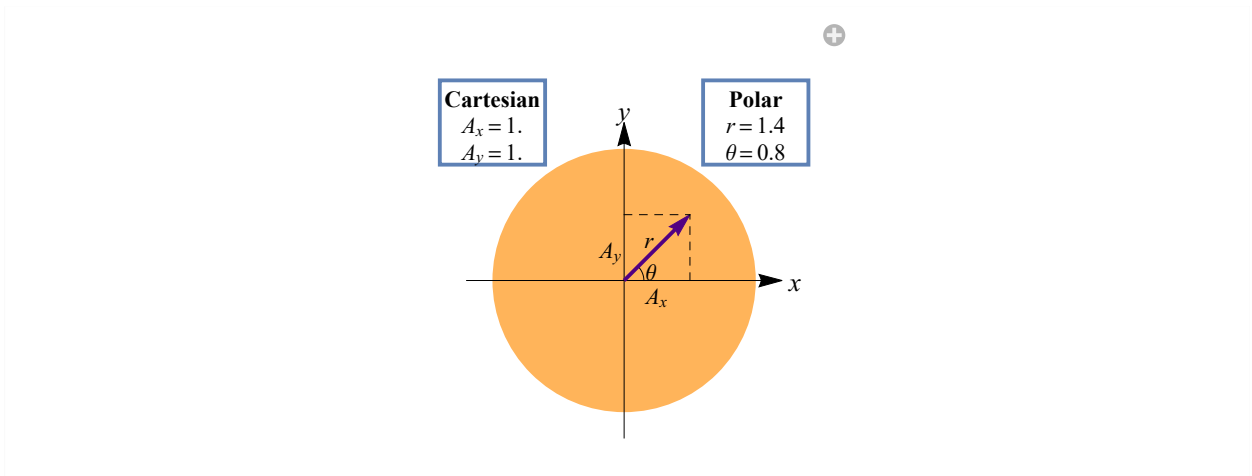
which in polar coordinates equals

$$\vec{A} = r \cos[\theta] \hat{x} + r \sin[\theta] \hat{y} \quad (2)$$

where

$$r = \sqrt{A_x^2 + A_y^2} \quad (3)$$

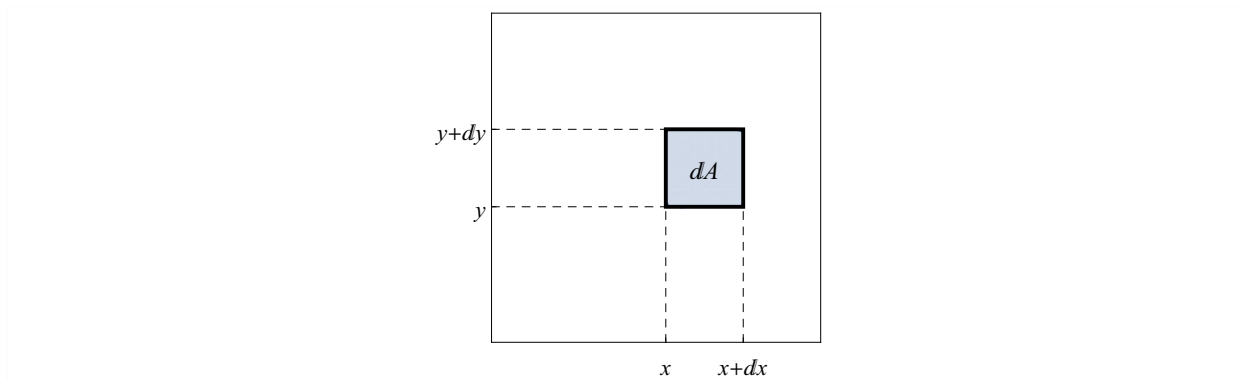
$$\theta = \text{ArcTan}\left[\frac{A_y}{A_x}\right] \quad (4)$$



When doing 2D integrals in Cartesian coordinates, we are familiar with the area element $dx dy$. We use this when computing an integral $\iint f[x, y] dx dy$. But where does $dx dy$ come from? If we consider a generic point (x, y)

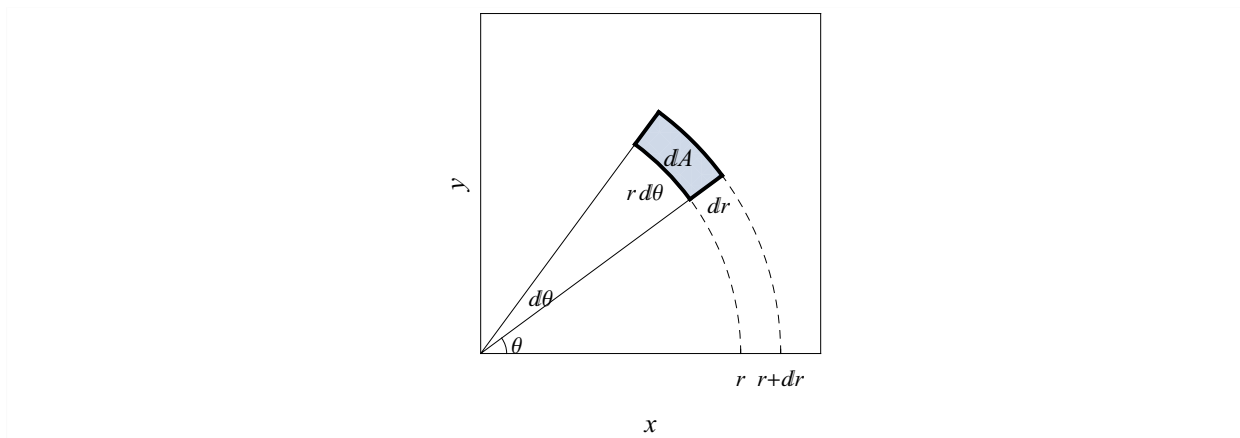
and infinitesimally displace it in the x -direction and y -direction by dx and dy , respectively, we find the points (x, y) , $(x + dx, y)$, $(x, y + dy)$, and $(x + dx, y + dy)$. What is the area of the region encompassed by these points? Simply $dx dy$!

Cartesian Coordinate Area Element $dA = dx dy$ (5)



The situation is the same for polar coordinates, although perhaps slightly less familiar. Consider a generic point (r, θ) and the infinitesimally displaced points found when we vary the first coordinate by dr and the second coordinate by $d\theta$. Thus we have four points (r, θ) , $(r + dr, \theta)$, $(r, \theta + d\theta)$, and $(r + dr, \theta + d\theta)$. As shown in the figure below, the element of this infinitesimal area equals $(dr)(r d\theta) = r dr d\theta$. Thus,

Polar Coordinate Area Element $dA = r dr d\theta$ (6)

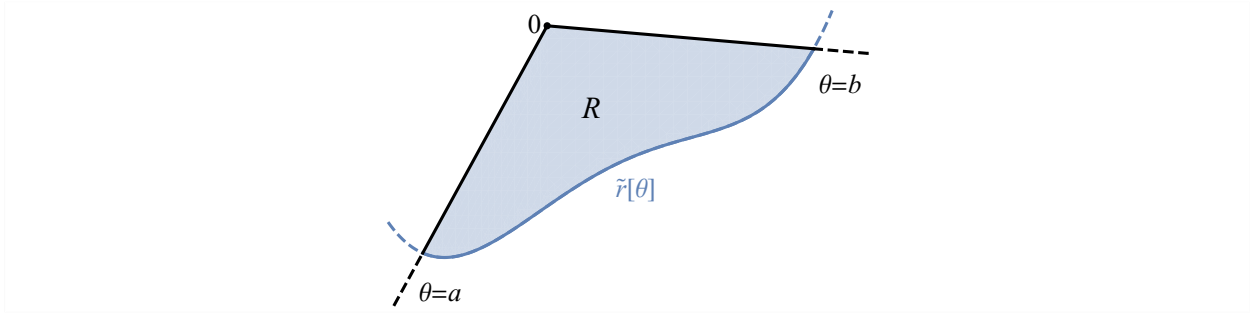


What does this imply? If we integrate $f[r, \theta]$ over a region in polar coordinates, then the integral must take the form

$$\iint f[r, \theta] r dr d\theta \quad (7)$$

Example

Compute the area of a region R enclosed by the curve $\tilde{r}[\theta]$ and the rays $\theta = a$ and $\theta = b$ where $0 < b - a \leq 2\pi$.

**Solution**

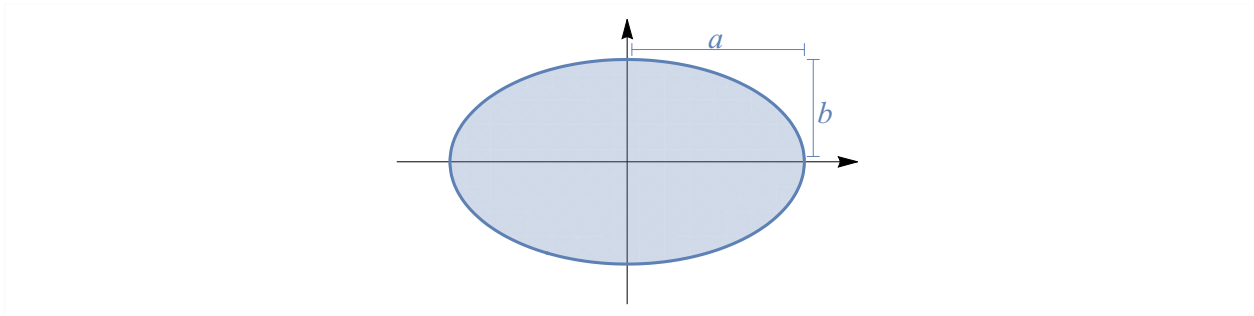
To compute an area, we set $f[r, \theta] = 1$ in Equation (7) and find the area

$$\int_a^b \int_0^{\tilde{r}[\theta]} r \, dr \, d\theta = \frac{1}{2} \int_a^b \tilde{r}[\theta]^2 \, d\theta \quad (8)$$

which is a famous formula for area in polar coordinates. \square

Example

Find the area of an ellipse with semi-major axis a and semi-minor axis b .

**Solution**

Orienting our ellipse as in the figure above, the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (9)$$

In polar coordinates ($x = r \cos[\theta]$, $y = r \sin[\theta]$) this becomes

$$\frac{r^2 \cos[\theta]^2}{a^2} + \frac{r^2 \sin[\theta]^2}{b^2} = 1 \quad (10)$$

Method 1: Integrate using Polar Coordinates

As shown in the previous Example, the area of the ellipse equals $\frac{1}{2} \int_0^{2\pi} \tilde{r}[\theta]^2 \, d\theta$ where $\tilde{r}[\theta]$ traces the boundary of the ellipse. From Equation (10),

$$\tilde{r}[\theta]^2 = \frac{1}{\frac{\cos[\theta]^2}{a^2} + \frac{\sin[\theta]^2}{b^2}} = \frac{a^2 b^2}{b^2 \cos[\theta]^2 + a^2 \sin[\theta]^2} \quad (11)$$

Integrating over the top-half of the ellipse and multiplying by 2 (for the bottom half), the area of the ellipse equals

$$\int_0^\pi \frac{a^2 b^2}{b^2 \cos[\theta]^2 + a^2 \sin[\theta]^2} \, d\theta = a b \operatorname{ArcTan} \left[\frac{a \sin[\theta]}{b \cos[\theta]} \right]_{\theta=0}^{\theta=\pi} = \pi a b - 0 = \pi a b \quad (12)$$

You must be very careful when taking the bounds on this integral, because of the branch cuts (i.e. the discontinuities) in ArcTan. When taking the bounds on this integral you must take the modulus of the ArcTan function so that it becomes continuous. Moral of the story: Check your answer with *Mathematica* (but note that even with *Mathematica* it is easy to get mixed up about the ArcTan branch cuts!)

Integrate $\left[\frac{a^2 b^2}{b^2 \cos[\theta]^2 + a^2 \sin[\theta]^2}, \{\theta, 0, \pi\}, \text{Assumptions} \rightarrow 0 < b < a \right]$

$a b \pi$

Method 2: Integrate using Cartesian Coordinates

Using the equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{13}$$

we can integrate over the top-half of the ellipse in Cartesian coordinates (multiplying by 2 to account for the bottom half)

$$2 \int_{-a}^a b \left(1 - \frac{x^2}{a^2}\right)^{1/2} dx = b \left\{ x \left(1 - \frac{x^2}{a^2}\right)^{1/2} + a \operatorname{ArcSin}\left[\frac{x}{a}\right] \right\}_{x=-a}^{x=a} = a b \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi a b \tag{14}$$

This integral is nasty, so you probably won't tackle the problem this way without *Mathematica* at your back. However, it is nice to check that this method yields the same answer.

Method 3: A Clever Trick

By far the easiest way to find the area of an ellipse is to note that the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{15}$$

becomes a circle if we stretch our coordinate system in the y -direction using the transformation $\tilde{y} = \frac{a}{b} y$,

$$\frac{x^2}{a^2} + \frac{\tilde{y}^2}{a^2} = 1 \tag{16}$$

This is the equation of a circle of radius a , and therefore of area πa^2 . An infinitesimal unit of area in our normal coordinates, $dx dy$, is related to the infinitesimal unit of area in the new coordinates, $dx d\tilde{y} = \frac{b}{a} dx d\tilde{y}$. In other words, any area that we measure in the x - y coordinate system is given by $\frac{a}{b}$ times the area measured in the x - \tilde{y} coordinate system, which equals $\frac{b}{a} (\pi a^2) = \pi a b$. \square

3D Spherical Coordinates

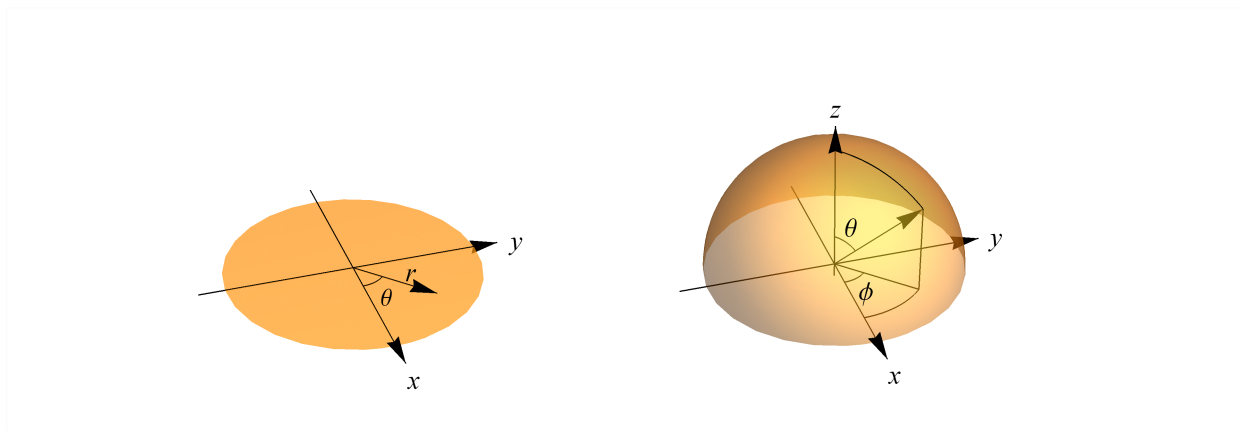
A generic 3D vector in Cartesian coordinates is

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \tag{17}$$

which in spherical polar coordinates is given by

$$\vec{A} = r \sin[\theta] \cos[\phi] \hat{x} + r \sin[\theta] \sin[\phi] \hat{y} + r \cos[\theta] \hat{z} \tag{18}$$

where $\phi \in [0, 2\pi)$ is the polar angle and $\theta \in [0, \pi]$ is the azimuthal angle.



Unfortunately, as seen above, physicists take the horrible convention where θ is the polar angle in 2D radial coordinates but ϕ is the polar angle in 3D spherical coordinates. Therefore, to regain polar coordinates from spherical coordinates you must set $\theta = \frac{\pi}{2}$ and then rename $\phi = \theta$. (Mathematicians, on the other hand, tend to switch the definitions of θ and ϕ in spherical coordinates to avoid this issue!)

We can convert from Cartesian coordinates to spherical coordinates using

$$r = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (19)$$

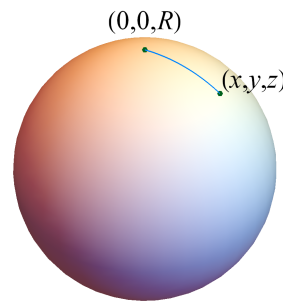
$$\theta = \text{ArcTan}\left[\frac{(A_x^2 + A_y^2)^{1/2}}{A_z}\right] = \text{ArcCos}\left[\frac{A_z}{(A_x^2 + A_y^2 + A_z^2)^{1/2}}\right] \quad (20)$$

$$\phi = \text{ArcTan}\left[\frac{A_y}{A_x}\right] \quad (21)$$

(I encourage you to verify that both of these expressions for θ are equal!)

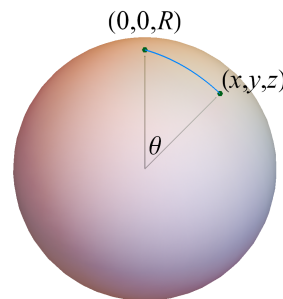
Example

Staying on a sphere of radius R , what is the shortest distance between the point $(0, 0, R)$ on the z -axis and a general point (x, y, z) on the sphere?



Solution

Rewrite the point (x, y, z) in spherical coordinates as (R, θ, ϕ) where $\theta = \text{ArcTan}\left[\frac{(x^2 + y^2)^{1/2}}{z}\right]$ and $\phi = \text{ArcTan}\left[\frac{y}{x}\right]$ are given by Equations (20) and (21). Note that the shortest distance on a sphere is given by an arc along a *great circle*, that is, a circle whose center lies at the center of the sphere. In this case, the arc can be parameterized in spherical coordinates as (R, θ', ϕ) where $\theta' \in [0, \theta]$ moves you from the point on the z -axis to the point of interest.



Therefore, the shortest distance between these two point while staying on the sphere is given by the length of this arc, which is

$$R \theta = R \text{ArcTan}\left[\frac{(x^2 + y^2)^{1/2}}{z}\right] \quad (22)$$

We can quickly check some limits. When $z = R$ and $x = y = 0$, we correctly find the distance to be zero. When $z = -R$ and $x = y = 0$, you have to be more careful since $\text{ArcTan}[0] = 0$, but this value is not unique since

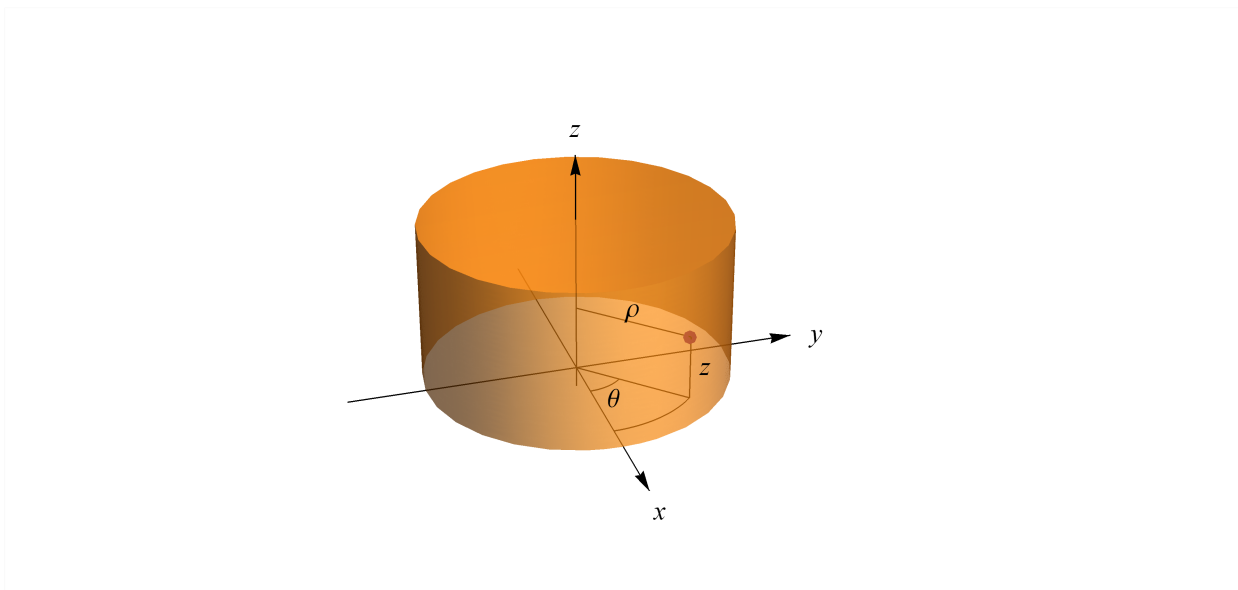
$\tan[0] = \tan[\pi] = 0$, and indeed in this case $\theta = \pi$. When $z = 0$ and you are anywhere on the $z = 0$ plane, the distance is given by $R \operatorname{ArcTan}[\infty] = \frac{R\pi}{2}$ which is one fourth the circumference of the sphere, as expected. \square

3D Cylindrical Coordinates

Another 3D coordinate system that is occasionally convenient is cylindrical coordinates which has a radial distance ρ , a polar angle θ , and a height component z . Essentially, cylindrical coordinates extend 2D polar coordinates to 3D by adding the Euclidean height z . A vector $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$ in Cartesian coordinates would be written in cylindrical coordinates as

$$\vec{A} = \rho \cos[\theta] \hat{x} + \rho \sin[\theta] \hat{y} + z \hat{z} \quad (23)$$

where $\theta \in [0, 2\pi)$ is the polar angle.



We can convert from Cartesian coordinates to cylindrical coordinates using

$$\rho = \sqrt{A_x^2 + A_y^2} \quad (24)$$

$$\theta = \operatorname{ArcTan}\left[\frac{A_y}{A_x}\right] \quad (25)$$

$$z = A_z \quad (26)$$

Mathematica Initialization